Weyl's wedges

1998 J. Phys. A: Math. Gen. 31 1911

(http://iopscience.iop.org/0305-4470/31/8/005)

J. Phys. A: Math. Gen. **31** (1998) 1911–1928. Printed in the UK PII: S0305-4470(98)89591-7

Weyl's wedges

C J Howls and S A Trasler

Department of Mathematics and Statistics, Brunel University, Uxbridge, Middlesex, UB8 3PH, UK

Received 28 November 1997

Abstract. The semiclassical Weyl series for an arbitrary-angle, zero-potential, circular-wedge quantum billiard with Dirichlet boundary conditions is derived. The goal is to study the effect of non-smooth boundaries on a conjecture of Berry and Howls (1994) concerning the high orders. The dominant behaviour of the late terms is identified, together with correction terms. The factorial-over-power and correction behaviour is found to be in accordance with an extension of the work of Berry and Howls. As might be expected, the only dominant contributions from the polygonal corner are to the 'length' and 'constant' terms of the Weyl series. The same is not true for the other angles. Surprisingly, only one periodic orbit arising from the wedge geometry affects the Weyl series for arbitrary angle of opening γ, although there is a subdominant residue from a memory of the circular symmetry. The prefactor of this residue is proportional to γ. Nevertheless, with one exception, the analytic behaviour of the Weyl series conspires to force the appearance of only the expected wedge periodic orbits in the exponential corrections.

1. Introduction

Berry and Howls (1994) (hereafter called BH) formulated a conjecture about the structure of the high orders of the semiclassical expansion of spectral functions pertaining to two-dimensional quantum billiards of area A with C^∞ boundaries. Specifically, the regularized resolvent (Voros 1992)

$$g(s) \equiv \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n+s^2}-\frac{A}{4\pi}\ln\left(\frac{E_N}{s^2}\right)\right] \tag{1}$$

was expanded in terms of the large energy variable $s = \mathrm{i}\sqrt{E}$ as its Weyl series

$$g(s) \sim \sum_{r=1}^{\infty}\frac{c_r}{s^r}. \tag{2}$$

(The periodic orbit corrections are exponentially small in this complex choice of scaled energy variable. Other spectral functions can be related to the regularized resolvent by transforms, Voros (1992).) Strong numerical evidence was given in several examples that

$$c_r \to \frac{\alpha(r-\beta)!}{l^r} \qquad r\to\infty \tag{3}$$

where l is the length of a periodic orbit associated with the classical billiard system. The conjecture was based on a formal Borel summation of the leading-order behaviour (2) and on formal resurgence results from exponential asymptotics (Dingle 1973, Voros 1983, Berry and Howls 1991, Howls 1992). When l was a two-bounce orbit BH suggested a graphical selection mechanism, based on local boundary coordinates. It transpired that l need not be the shortest, most stable or even a real orbit. They also discussed several classes of counterexamples.

The parameters α and β were identified by comparison with the prefactor of the relevant exponential periodic orbit contribution. Voros (1983) derived similar results for classes of one-dimensional Schrödinger potentials, but his approach was based on the more complete theory of resurgence (Ecalle 1981, 1984).

The reasons for considering the high orders go beyond pure mathematical interest (Levitin 1997). The coefficients c_r give values of spectral zeta functions at negative integer values, which are important in field theoretic problems (Bordag *et al* 1996a, b, Elizalde *et al* 1993, Lesduarte and Romeo 1994). Furthermore, if they can be calculated, the late terms can provide an algebraic tool for deducing the local Riemann sheet structure of the expansion for the resolvent, using a method outlined by Howls (1997). In turn, this can lead to the identification of approximate functional equations satisfied by the spectral function in question (Voros 1992, Howls and Trasler 1998). The latter technique is an extended corollary of Voros (1983) and will be discussed elsewhere in the context of billiard systems.

In this paper we wish to study the conjecture for a billiard with a non-analytic boundary. We choose to study a class of 'cake wedge' billiards without potentials, being angular sectors of a unit radius circular billiard, satisfying (figure 1)

$$(-\nabla^2 + s^2)\psi(\boldsymbol{r}) = 0 \quad \boldsymbol{r} \in \Omega \qquad \text{with} \qquad \psi(\boldsymbol{r}) = 0 \quad \boldsymbol{r} \in \partial\Omega. \tag{4}$$

We study all angles of opening $0 < \gamma < 2\pi$. (Dietz *et al* 1995 examined the relationship between exterior and interior problems for such wedges.) Due to the high degree of symmetry we do not claim that the results will be general, although they do appear to confirm that a form of the conjecture may still hold for non-polygonal billiards with corner contributions. In addition, subdominant contributions can be identified. Hence a more complete conjecture† can be posed for the wedge billiard:

$$c_r \sim \sum_j \sum_{k=0}^{\infty} \frac{\alpha_k^{(j)}(r-k-\beta_j)!}{l_j^{r-k}} \qquad r \to \infty. \tag{5}$$

The j-sum is over a (formally possibly infinite) set of periodic orbits l_j *associated* with the classical system. (They may not in fact be *actual* periodic orbits of the classical system itself, cf BH section 2c.) The parameters characterize the periodic orbit l_j. A formal Borel summation of the tail of each k-series in the j-sum will generate a periodic-orbit-like contribution according to BH section 1. This conjecture is examined in the context of general billiard systems elsewhere (Howls and Trasler 1998). In section 2 we derive the regularized resolvent of the cake wedge, before expanding out the Weyl series. In section 3 the coefficients c_r are calculated. The dominant form of the later c_r is identified and higher-order corrections obtained. An analysis of other periodic orbit contributions to the Weyl series follows in section 4. We conclude in section 5 with a discussion.

2. Calculation of the resolvent and Weyl series

For real energies E, the eigenfunctions of the wedge are

$$\psi_{mn}(r,\theta) = A_{mn} J_{m\pi/\gamma}(r\sqrt{E_{mn}}) \sin(m\pi\theta/\gamma) \tag{6}$$

† It could be argued that an expansion in $1/r$ would be more appropriate asymptotically. This is a common form in field theory (Le Guillou and Zinn-Justin 1990). However, from results in resurgence (Dingle 1973, Balian *et al* 1979, Berry and Howls 1991) it transpires that *inverse factorial series* such as (5) are more intimately linked to the analytic structure of the resolvent via the expansion in $1/s$.

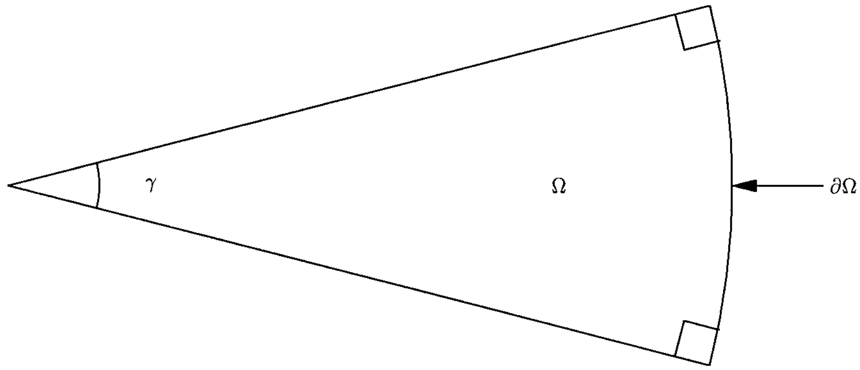

Figure 1. The wedge billiard Ω with sectorial angle γ and two right angles between the line segments and the arc.

with eigenvalues $E_{mn} = {j_{mn}^{(\gamma)}}^2$, where $j_{mn}^{(\gamma)}$ is the nth zero of the Bessel function $J_{m\pi/\gamma}$. As in BH we shall work with complex energies s, the better for resolving the exponential scales in the problem.

The eigenfunctions are known explicitly, so it is possible to use the Mellin–Barnes techniques of Bordag *et al* (1996c) to calculate the required expansion. However, we choose to use a Poisson summation method for two reasons. First, we find this approach more favourable for comparison with the techniques and the results of BH section 3, which allows for a clearer identification of the geometric origin of each contribution. Secondly, due to the presence of corners, it illustrates the necessity for including all harmonics of the Poisson sum, in order to resolve the corner contributions. The latter point is salutary, since semiclassical expanders often neglect higher harmonics when seeking average properties, even with C^∞ boundaries, which can lead to great confusion (Waechter 1972, Kennedy 1978, Bordag *et al* 1996c).

We follow Stewartson and Waechter (1971) and BH by solving

$$(-\nabla^2 + s^2)G(\boldsymbol{r}, \boldsymbol{r}_0, s) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) \qquad \boldsymbol{r}, \boldsymbol{r}_0 \in \Omega \tag{7}$$

for the Green function G, subject to the conditions

$$G(\boldsymbol{r}, \boldsymbol{r}_0, s) = 0 \text{ when } \begin{cases} r \ (r_0) = 1 & 0 \leqslant \theta \ (\theta_0) \leqslant \gamma \\ \theta \ (\theta_0) = 0, \gamma & 0 \leqslant r \ (r_0) \leqslant 1. \end{cases} \tag{8}$$

The resolvent is then the trace

$$g(s) = \int_\Omega \mathrm{d}\boldsymbol{r} \lim_{\boldsymbol{r}_0 \to \boldsymbol{r}} \chi \qquad G = G_0 + \chi. \tag{9}$$

Here G_0 is the free-space Green function,

$$G_0 = \frac{1}{2\pi} K_0(s|\boldsymbol{r} - \boldsymbol{r}_0|) \tag{10}$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} I_m(sr) K_m(sr_0)\, \mathrm{e}^{\mathrm{i}m(\theta - \theta_0)} \qquad r < r_0 \tag{11}$$

which satisfies (7) but not the boundary condition. The function χ is a solution of the homogeneous equivalent of (7) and cancels out G_0 on the boundary. A Fourier representation of the total Green function in the wedge is given by

$$G(\boldsymbol{r}, \boldsymbol{r}_0, s) = \frac{2}{\gamma} \sum_{n=1}^{\infty} a_n(r, r_0, s) \sin \frac{n\pi\theta}{\gamma} \sin \frac{n\pi\theta_0}{\gamma} \tag{12}$$

where the coefficients $a_n(r, r_0, s)$ satisfy the top two conditions of (8). A short calculation gives

$$G(\boldsymbol{r}, \boldsymbol{r}_0, s) = \frac{1}{\gamma} \sum_{n=-\infty}^{\infty} I_{|n|\pi/\gamma}(sr) \left\{ K_{|n|\pi/\gamma}(sr_0) - \frac{K_{|n|\pi/\gamma}(s)}{I_{|n|\pi/\gamma}(s)} I_{|n|\pi/\gamma}(sr_0) \right\} \times \sin \frac{n\pi\theta}{\gamma} \sin \frac{n\pi\theta_0}{\gamma} \tag{13}$$

where $r < r_0$. Note that for technical reasons we have extended the result so that the n-sum runs over all integer values. Hence we have

$$\lim_{r\to r_0} \chi = \frac{1}{\gamma} \sum_{n=-\infty}^{\infty} I_{|n|\pi/\gamma}(sr) \left\{ K_{|n|\pi/\gamma}(sr) - \frac{K_{|n|\pi/\gamma}(s)}{I_{|n|\pi/\gamma}(s)} I_{|n|\pi/\gamma}(sr) \right\} \sin^2 \frac{n\pi\theta}{\gamma} - \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} I_m(sr) K_m(sr). \tag{14}$$

To find the resolvent we need to integrate (14) over the billiard. The term in the square of the Bessel function I presents no problems, but the product of I with K needs greater attention. We want to evaluate (Watson 1948)

$$\int \mathrm{d}r\, r I_m(sr) K_m(sr) = \frac{r^2}{2} \left\{ \left(1 + \frac{m^2}{(sr)^2}\right) I_m(sr) K_m(sr) - I_m'(sr) K_m'(sr) \right\} \tag{15}$$

over $0 \leqslant r \leqslant 1$. The value of the lower limit is not immediately obvious: we use the small-argument asymptotics of the Bessel functions (Olver 1974) to examine it. For small z and $p \geqslant 0$ (the functions are even with respect to integer order),

$$I_p(z) \sim \frac{(z/2)^p}{p!} \tag{16}$$

$$K_p(z) \sim \frac{(p-1)!}{2(z/2)^p} \qquad K_0(z) \sim -\ln z. \tag{17}$$

Applying these facts with (15) in mind, we have the following limits:

$$\lim_{r\to 0} r^2 I_p(sr) K_p(sr) = 0 \tag{18}$$

$$\lim_{r\to 0} \frac{p^2}{s^2} I_p(sr) K_p(sr) = \frac{|p|}{s^2} \tag{19}$$

$$\lim_{r\to 0} r^2 I_p'(sr) K_p'(sr) = -\frac{|p|}{s^2}. \tag{20}$$

(Note that in the above, if p is negative, it must be an integer. However, this problem is circumvented in what follows.) The radial integrals are therefore

$$\int_0^1 \mathrm{d}r\, r I_m(sr) K_m(sr) = \frac{1}{2} \left\{ \left(1 + \frac{m^2}{s^2}\right) I_m(s) K_m(s) - I_m'(s) K_m'(s) - \frac{|m|}{s^2} \right\} \tag{21}$$

$$\int_0^1 \mathrm{d}r\, r I_m^2(sr) = \frac{1}{2} \left\{ \left(1 + \frac{m^2}{s^2}\right) I_m(s) K_m(s) - I_m'(s) K_m'(s) - \frac{I_m'(s)}{s I_m(s)} \right\}. \tag{22}$$

The angular integration (over $0 \leqslant \theta \leqslant \gamma$) is trivial and we obtain

$$g(s) = \frac{1}{4}\sum_{n=-\infty}^{\infty}(1-\delta_{n0})\,\{f^*_{|n|\pi/\gamma}(s) - f_{|n|\pi/\gamma}(s)\} - \frac{\gamma}{4\pi}\sum_{m=-\infty}^{\infty} f^*_m(s) \tag{23}$$

where (cf BH equation (54))

$$f_p(s) = \left(1+\frac{p^2}{s^2}\right) I_p(s)K_p(sr) - I'_p(sr)K'_p(sr) - \frac{I'_p(s)}{sI_p(s)} \tag{24}$$

$$f^*_p(s) = f_p(s) + \frac{I'_p(s)}{sI_p(s)} - \frac{|p|}{s^2} \tag{25}$$

and δ_{pq} is the Kronecker delta. This may be rearranged to give

$$g(s) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\{f^*_{|n|\pi/\gamma}(s) - f_{|n|\pi/\gamma}(s)\} - \frac{\gamma}{4\pi}\sum_{m=-\infty}^{\infty} f^*_m(s) - \frac{I'_0(s)}{4sI_0(s)}. \tag{26}$$

To obtain a computable expansion, we apply the Poisson sum formula (BH equation (53))

$$\sum_{n=-\infty}^{\infty} h_n(s) = \sum_{\mu=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}n\, h_n(s)\,\mathrm{e}^{2\pi\mathrm{i}n\mu} \tag{27}$$

to (23), after extracting the Kronecker delta. A few manipulations reveal

$$g(s) = \frac{\gamma}{2\pi}\sum_{\mu=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}m\{[f^*_m(s) - f_m(s)]\,\mathrm{e}^{2\gamma\mathrm{i}m\mu} - f^*_m(s)\,\mathrm{e}^{2\pi\mathrm{i}m\mu}\} - \frac{I'_0(s)}{4sI_0(s)}. \tag{28}$$

This can be decomposed into three parts

$$g(s) = \mathrm{A}+\mathrm{B}+\mathrm{C} \tag{29}$$

where

$$\mathrm{A} = -\frac{I'_0(s)}{4sI_0(s)} \tag{30}$$

$$\mathrm{B} = \frac{\gamma}{\pi}\sum_{\mu=1}^{\infty}\int_0^{\infty}\mathrm{d}m\left\{\left(\frac{I'_m(s)}{sI_m(s)} - \frac{m}{s^2}\right)\cos 2\gamma m\mu - f^*_m(s)\cos 2\pi m\mu\right\} \tag{31}$$

$$\mathrm{C} = -\frac{\gamma}{2\pi}\int_0^{\infty}\mathrm{d}m\; f_m(s). \tag{32}$$

We identify C as a fraction $(\gamma/2\pi)$ of the *algebraic part* of the resolvent (i.e. the Weyl series) for the circle billiard (cf BH section 3 equations (53) and (54)), a fact we will exploit in our calculation of the coefficients for the wedge. After Poisson summation, the terms involving the products $I_m(s)K_m(s)$ and $I'_m(s)K'_m(s)$ only contribute to c_2. This fact was not pointed out by BH.

To identify the role of the term B we expand the Bessel functions to leading order in $1/s$ (Abramowitz and Stegun 1972):

$$\frac{I'_m(s)}{sI_m(s)} \sim \frac{\sqrt{m^2+s^2}}{s^2}. \tag{33}$$

Making the substitution $xs = m$, integrating twice by parts and using the identity (Gradshteyn and Ryzhik 1965)

$$\int_0^{\infty}\mathrm{d}x\,\frac{\cos qx}{(1+x^2)^{3/2}} = qK_1(q) \tag{34}$$

$$\Rightarrow \qquad \int_0^{\infty}\mathrm{d}x(\sqrt{1+x^2}-x)\cos qx = \frac{1}{q^2} - \frac{K_1(q)}{q} \tag{35}$$

we have B in the form

$$\mathrm{B} \sim \frac{\gamma}{\pi}\sum_{\mu=1}^{\infty}\left\{\left(\frac{1}{(2\gamma\mu s)^2}-\frac{1}{(2\pi\mu s)^2}\right)-\left(\frac{K_1(2\gamma\mu s)}{2\gamma\mu s}-\frac{K_1(2\pi\mu s)}{2\pi\mu s}\right)\right\}+\cdots \tag{36}$$

$$=\frac{\pi^2-\gamma^2}{24\pi\gamma}-\frac{\gamma}{\pi}\sum_{\mu=1}^{\infty}\left\{\frac{K_1(2\gamma\mu s)}{2\gamma\mu s}-\frac{K_1(2\pi\mu s)}{2\pi\mu s}\right\}+\cdots. \tag{37}$$

A short calculation (appendix A) shows the Bessel series give a contribution at $\mathrm{O}(\mathrm{e}^{-ls}s^{-3/2})$, $l = 2\pi, 2\gamma$, and so they do not contribute to the Weyl series.

Term A is evaluated by replacing the Bessel functions by their asymptotic expansions for large arguments (appendix B), leading to a formal power series in $1/s$.

In total, the first two terms of the Weyl series are then

$$g(s) = -\frac{2+\gamma}{8s}+\left\{\frac{\gamma}{12\pi}+\frac{1}{8}+\frac{\pi^2-\gamma^2}{24\pi\gamma}\right\}\frac{1}{s^2}+\mathrm{O}\left(\frac{1}{s^3}\right). \tag{38}$$

The results of McKean and Singer (1967), Stewartson and Waechter (1971) state that for a two-dimensional billiard with a piecewise smooth Dirichlet boundary of length L, curvature $\kappa(\sigma)$ and corners of internal angle γ_i,

$$g(s) = -\frac{L}{8s}+\left\{\frac{1}{12\pi}\oint \mathrm{d}\sigma\,\kappa(\sigma)+\sum_i c(\gamma_i)\right\}\frac{1}{s^2}+\mathrm{O}\left(\frac{1}{s^3}\right) \tag{39}$$

with the corner contributions given by

$$c(\eta) = \frac{\pi^2-\eta^2}{24\pi\eta} \qquad 0<\eta<2\pi. \tag{40}$$

To reconcile (38) and (39), we observe that the perimeter of the wedge has length $L = 2+\gamma$ and internal angles $\gamma_0 = \gamma$, $\gamma_1 = \gamma_2 = \pi/2$. The curvature integral vanishes on the radial portions.

It is clear that there are natural geometrical interpretations for the three contributions to c_r from (29). Identifying the features of the boundary, namely the line segments, arc, polygonal and other corners, we can trace their appearances in the highly-geometrical early terms of (38) back to the components A, B and C.

Term C is an echo of the circular symmetry in the wedge, describing the curved section of the boundary and the area of the shape. Further information about the sectorial angle γ appears through term B, although it is only relevant to c_2 in the Weyl series. Finally, the radial lengths (seen in c_1) and the right angles (c_2) are accounted for in term A.

We might have expected γ not to appear in the late terms, since the wedge is locally polygonal there and Weyl series truncate at finite orders for polygons (Baltes and Hilf 1976). In fact it does influences the high c_r, but only through a 'false' subdominant memory of the circle.

3. Preliminary results

Formally expanding terms A and C as power series in $1/s$ (appendix B, BH equations (55)–(58)) and adding in term B, we obtain the coefficients c_r for arbitrary γ using *Mathematica*. The first few coefficients are shown in table 1.

First we restrict our attention to the dominant form of the conjecture (3), (cf (5)) and obtain numerical estimates of the relevant α, β and l. Subdominant contributions to the full conjecture (5) from other periodic orbits are considered in section 4.

Table 1. The first ten coefficients of the Weyl series for the wedge billiard with sectorial angle γ and Dirichlet boundary.

r	c_r
1	$-\frac{2+\gamma}{8}$
2	$\frac{3\pi+2\gamma}{24\pi}$
3	$\frac{16+\gamma}{512}$
4	$\frac{315\pi+32\gamma}{10080\pi}$
5	$\frac{6400+111\gamma}{131072}$
6	$\frac{585\,585\pi+17408\gamma}{576\,5760\pi}$
7	$\frac{1\,098\,752+5705\gamma}{4\,194\,304}$
8	$\frac{499\,534\,035\pi+4\,468\,736\gamma}{620\,780\,160\pi}$
9	$\frac{24\,624\,037\,888+38\,306\,807\gamma}{8\,589\,934\,592}$
10	$\frac{1\,405\,307\,919\,015\pi+3\,731\,881\,984\gamma}{120\,942\,581\,760\pi}$

A strictly numerical estimate for α is only possible with prior knowledge of β and l. To that end, if (3) holds then, for a specific large value of r,

$$\tau(r) \equiv \frac{c_r c_{r-2}}{c_{r-1}^2} \sim \frac{r-\beta}{r-\beta-1} \tag{41}$$

and hence we obtain the approximation

$$\beta = \tau/(\tau-1) + \mathrm{O}(1/r). \tag{42}$$

A graph of $\tau/(\tau-1)$ against r should therefore asymptote to the value β. This is shown in figure 2 for several choices of γ. Numerically we see that in all the graphs $\beta \simeq 2$.

The equality $\beta = 2$ is justified analytically later. Assuming this result, we can find l by considering the slope of the function

$$-\ln\left(\frac{c_r}{(r-2)!}\right) \sim r\ln l - \ln\alpha \tag{43}$$

plotted against r. This is plotted in figure 3 for a variety of angles. It emerges that $l \simeq 2$, again apparently independent of γ. The length 2 corresponds to an orbit which bounces once on the circular boundary and once in the corner of angle γ.

We obtain an approximation for α by plotting $2^r c_r/(r-2)!$ against r. Figure 4 shows graphs of this function: we find $\alpha \simeq 0.317$, once more independent of γ and approximately equal to $1/\pi$.

It appears that the leading-order, late-term behaviour of the Weyl coefficients is insensitive to the sectorial angle. Closer examination of the resolvent shows this must be the case.

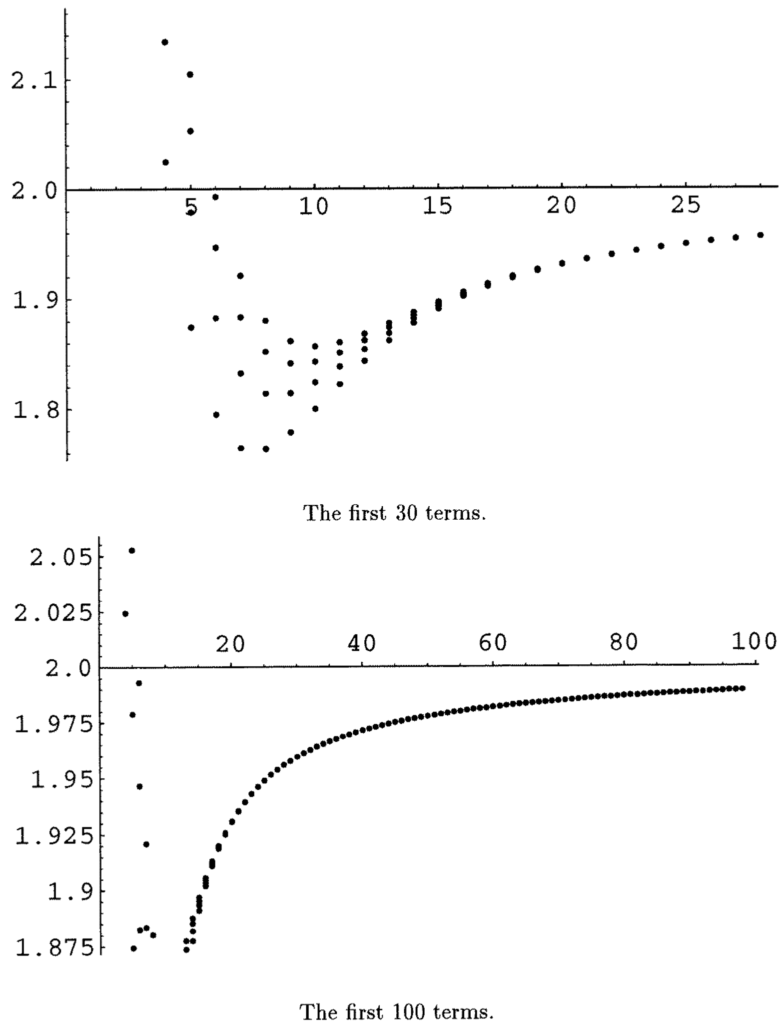

Figure 2. The value of the function in (42) against r for sectorial angles $\gamma = \pi/6$ (bottom set of points), 2, π and 4 (top). The difference at high orders is so slight that a graph of the first 30 terms is given by way of magnification.

We have already observed that term C reduces to a fraction of the (algebraic) Weyl series for the circle billiard. BH demonstrated that, to leading order, the coefficients there followed (3) with $l = 4$, while for the wedge billiards we have $l = 2$. Moreover, term C is γ-dependent. These two pieces of evidence imply that term C is not the controlling factor in high orders of the wedge Weyl series.

Term B does not affect the late terms and so term A, the ratio of Bessel functions, must be responsible for the behaviour we have found above. Therefore we isolate term A and attempt to justify the conjecture (5) for $l = 2$, by estimating some of the higher $\alpha_k^{(2)}$. Henceforth we will use the label $j = 2$ to refer to this (dominating) orbit and identify $\alpha_0^{(2)}$, β_2 and l_2 with the α, β and l respectively of (3).

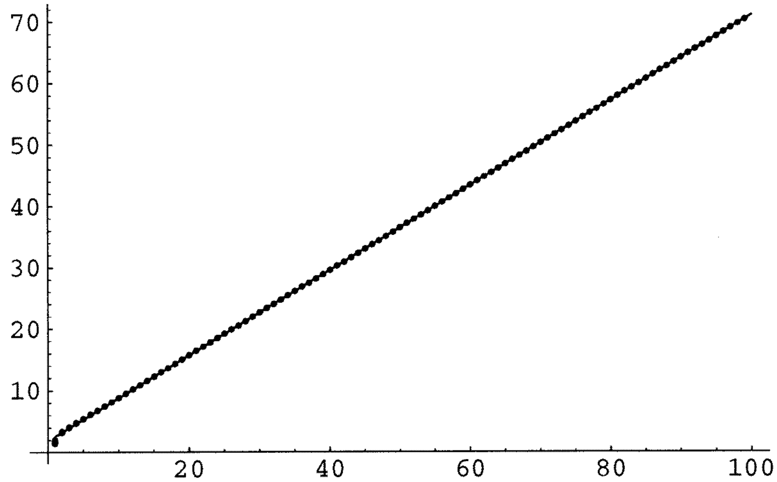

Figure 3. The function (43) against r for $\gamma = \pi/6$, 2, π and 4. A straight line of slope corresponding to $l = 2$ has been superimposed to obviate the agreement.

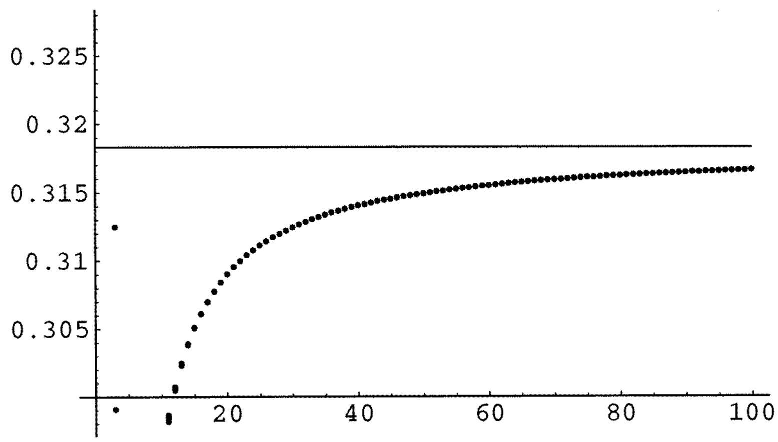

Figure 4. Crude estimates for α are obtained simply by accounting for the other (factorial and power) behaviour in the late-terms conjecture (3). The straight line corresponds to $\alpha = 1/\pi$. Plotted for $\gamma = \pi/6$, 2, π and 4.

Assuming form (5), we follow Voros (1983, appendix A) and use a Neville table procedure to estimate $\alpha_k^{(2)}$ and β_2 (appendix C). The results are presented in table 2. The values obtained for the first four terms, $\alpha_0^{(2)}$ to $\alpha_3^{(2)}$, are extremely close to fractions of $1/\pi$, which suggests that

$$c_r \sim \frac{1}{\pi}\frac{(r-2)!}{2^r} - \frac{1}{4\pi}\frac{(r-3)!}{2^{r-1}} - \frac{3}{32\pi}\frac{(r-4)!}{2^{r-2}} - \frac{13}{128\pi}\frac{(r-5)!}{2^{r-3}} - \cdots. \tag{44}$$

This result can be justified analytically from a direct expansion of the recurrence relation (B.4). This costly calculation is carried out in appendix D.

Table 2. The Neville table producing values for the first four $\alpha_k^{(2)}$. Errors are still small even at the fourth level of the iteration. The values are found to be approximately equal to $1/\pi$, $-1/4\pi$, $-3/32\pi$ and $-13/128\pi$, reading across the bottom of the table.

p	$\alpha_0^{(2)}$	$\alpha_1^{(2)}$	$\alpha_2^{(2)}$	$\alpha_3^{(2)}$
1	0.316 673 006	−0.080 207 151	−0.030 539 432	−0.033 498 262
2	0.318 323 322	−0.079 562 229	−0.029 815 409	−0.032 269 575
3	0.318 309 549	−0.079 578 063	−0.029 842 913	−0.032 332 215
4	0.318 309 900	−0.079 577 440	−0.029 841 459	−0.032 328 042
5	0.318 309 885	−0.079 577 474	−0.029 841 560	−0.032 328 396
6	0.318 309 886	−0.079 577 471	−0.029 841 552	−0.032 328 360
7	0.318 309 886	−0.079 577 472	−0.029 841 552	−0.032 328 365
8	0.318 309 886	−0.079 577 472	−0.029 841 552	−0.032 328 365
9	0.318 309 886	−0.079 577 472	−0.029 841 552	−0.032 328 365
10	0.318 309 886	−0.079 577 472	−0.029 841 552	
11	0.318 309 886	−0.079 577 472		
12	0.318 309 886			

4. Other orbit contributions to the Weyl series

In this section we investigate other contributions l_j to (5). First we recall some information about the circle.

Truncating the Weyl series at the least term r^* and Borel-summing the late terms, BH wrote the remainder in terms of an error function:

$$R(s) \simeq \frac{\mathrm{i}\pi\alpha}{(ls)^{\beta-1}} \mathrm{e}^{-ls} \operatorname{erf}\left(\frac{\mathrm{i}(r^* - ls)}{\sqrt{2r^*}}\right). \tag{45}$$

To find the prefactor α in (3), they exploited the Stokes phenomenon experienced by the last term (alone) in $f_m(s)$ as s is rotated in the complex plane to real values of energy E (Berry 1989). The leading exponential part of the resolvent is equated to (45) for $|\arg s| = \pi/2$ (as the error function tends to unity).

Using the Debye expansions for the Bessel functions, they wrote down (BH equation (61))

$$f_m^{(\exp)}(s) \simeq -\frac{2\sqrt{m^2+s^2}}{s^2} \sum_{p=1}^{\infty} (-\mathrm{i})^p (-1)^{mp}\, \mathrm{e}^{-pF} \tag{46}$$

$$F(s,m) = 2\sqrt{m^2+s^2} + 2m \ln\left(\frac{s}{m+\sqrt{m^2+s^2}}\right). \tag{47}$$

The term in $f_m(s)$ was then substituted in the Poisson-summed expression for the resolvent. Applying the method of steepest descent to the subsequent integrals in $g(s)$, the saddle points are given by

$$\arccos \frac{m}{\sqrt{E}} = \frac{\pi\mu}{p}. \tag{48}$$

This corresponds to the angle between a chord of a periodic orbit and the tangent to the circle where the chord cuts the boundary. The parameters p and μ are the number of bounces and turns about the origin, respectively, of an orbit of length

$$L_{\mathrm{c}} = 2p \sin \frac{\pi\mu}{p}. \tag{49}$$

With this in mind, we consider the wedge resolvent (29) term-by-term, at exponential order.

Term A undergoes a Stokes phenomenon which generates a contribution of the form (46) with $m = 0$

$$g^{(\mathrm{exp})}(s) \sim \frac{\mathrm{i}}{2s}\,\mathrm{e}^{-2s}. \tag{50}$$

This is entirely consistent with a Borel summation of the leading term of (44), thereby providing a check on the dominant form of c_r.

The only Stokes phenomenon which does not cancel in the second term (B) arises from the integrand of

$$\frac{\gamma}{\pi}\sum_{\mu=1}^{\infty}\int_0^{\infty}\mathrm{d}m\,\frac{I_m'(s)}{sI_m(s)}\cos 2\gamma m\mu. \tag{51}$$

The contribution to the wedge resolvent from this is (cf (46) and BH equation (62))

$$g^{(\mathrm{exp})}(s) = \frac{\gamma}{\pi}\sum_{p=1}^{\infty}(-\mathrm{i})^p\sum_{\mu=1}^{\infty}\int_0^{\infty}\mathrm{d}m\,\frac{\sqrt{m^2+s^2}}{s^2}\,\mathrm{e}^{\mathrm{i}m(\pi p-2\gamma\mu)-pF(s,m)}. \tag{52}$$

The integrand has saddles at positions given by

$$\arccos\frac{m}{\sqrt{E}} = \frac{\gamma\mu}{p} \tag{53}$$

(cf (48)). The exponent for these values of m is $-sL_{\mathrm{w}}$ where we have defined (cf (49))

$$L_{\mathrm{w}} = 2p\sin\frac{\gamma\mu}{p}. \tag{54}$$

Here μ has a quite different interpretation to that of the circle. While p still counts the number of bounces off the curved ('circular') part of the boundary, this μ is half the (always even) number of bounces off the straight edges of the wedge. (A corner collision counts as both a straight edge and an arc bounce.) A selection of orbits in the wedge billiard is shown in figure 5. Note that each orbit in the wedge only physically exists up to an associated angle γ.

The length-2 orbit is special: it is not described in the same terms as the others. There are shorter orbits (e.g. when $p = 1$), but they cannot be present in the Weyl series for two reasons. First, if they were, they would dominate the high c_r. Second, the algebraic form of B (37) shows that none of the (p,μ) orbits contribute to the late terms.

The exponential arising from a Stokes phenomenon in s due to a (p,μ) orbit is

$$\frac{\gamma(-\mathrm{i})^{p+1}L_{\mathrm{w}}^{3/2}}{4p^2\sqrt{2\pi s}}\,\mathrm{e}^{-sL_{\mathrm{w}}} \tag{55}$$

where we need $\pi p \geqslant 2\gamma\mu$ for the orbit to exist. It is easy to see that each (p,μ) combination corresponds to a one-parameter family of orbits. For an ordinary stationary path (osp), the following proportionality holds (Balian and Bloch 1972, p 101, cf p 153):

$$g^{(\mathrm{exp})}_{\mathrm{osp}}(s) \propto s^{q_0/2-1}\,\mathrm{e}^{-sL}. \tag{56}$$

Here the degeneracy factor $q_0 = 1$ gives the required $s^{-1/2}$ behaviour in (55).

Similar analysis of (50) reveals that, for $\gamma \neq \pi$, there is a one-parameter family of 2-bounce orbits between the polygonal corner and the arc. This can be understood by recalling that, for an orbit which encounters a singularity in the boundary of order ζ,

$$g^{(\mathrm{exp})}_{\mathrm{sing}}(s) \propto s^{(q_0-1)/2-\zeta}\mathrm{e}^{-sL} \tag{57}$$

(Balian and Bloch 1972, p 103, cf p 153). Here the polygonal corner gives $\zeta = 1$, so q_0 must be 1 to generate the s^{-1}. (When $\gamma = \pi$ this orbit joins two smooth sections of the

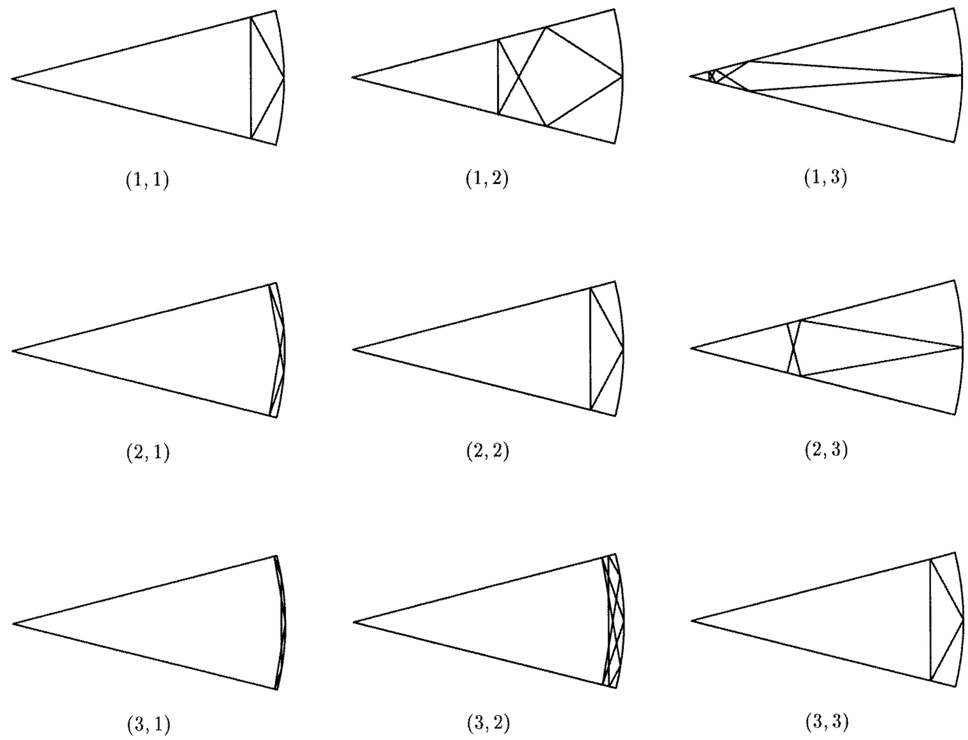

Figure 5. A selection of orbits in the wedge billiard, for $\gamma = 1/2$. We use the notation (p, μ), where p is the number of bounces off the curved edge and μ half the number off the straight edges.

boundary and is isolated, so for this angle only we use (56) with $q_0 = 0$ to maintain the s^{-1} of (50).)

Therefore, we note that the algebraic order of the former of these two exponential terms, at $O(s^{-1/2})$, is larger than the latter, at $O(s^{-1})$. That is, the most dominant orbits in the Weyl series and the periodic orbit corrections are not necessarily the same.

There is a further exponentially-small contribution from the higher harmonics of B, discarded as irrelevant to the Weyl series (see appendix A). These relate to orbits of lengths 2γ and 2π (and their repeats). The former is certainly the length of a whispering-gallery mode ($p \to \infty, \mu = 1$) for the wedge. The latter may be a memory of the circular symmetry. We do not explore this orbit further.

Term C contains none of the higher harmonics of the Poisson sum, so it does not generate any exponential circular periodic orbit contributions. However, it does undergo a Stokes phenomenon but, since the relevant saddle in the expansion of (52) lies at $m = -s^2$, to leading order the integral (with $\mu = 0$) evaluates to zero. Consequently, with the exception of the 2π-orbit above, any suggestion that the wedge *really* remembers its circular heritage at exponential (periodic orbit) order appears to be destroyed.

The situation is summarized in table 3. Perversely, the only physical wedge orbit that contributes to the Weyl series is the radial one with length 2. Indeed, this appears to be the only orbit present at both algebraic and exponential scales. Subdominant algebraic contributions to (5) arise from (longer) circle orbits. (Note that when γ is a rational fraction of π, these could be interpreted as repetitions of the shorter wedge orbits.)

Table 3. Contributions to the resolvent at different orders from different orbit classes.

Length	Algebraic	Exponential (via Stokes)
$L_w = 2p\sin(\gamma\mu/p)$		✓
$L_c = 2p\sin(\pi\mu/p)$	✓	
2	✓	✓

It is interesting to note what happens when $\gamma = \pi$ (the semicircle). The terms B and C combine to produce exactly half the full circle result while A remains undisturbed to account for the length-2 orbit. This makes sense: a semicircle has (obviously) half the area of a circle, manifested in the algebraic term c_1 of the Weyl series. In addition, the families of orbits (seen in the exponentials) are the same, bar the (extra) length-2 orbit, because the straight edge in the former is a line of symmetry in the latter.

The 2-bounce orbit (term A) survives as γ increases through π, becoming a diffractive orbit which scatters from the polygonal corner. As γ approaches 2π the circle result is not recovered. This can been seen most easily from the example in figure 6. There is no $(3, 1)$ orbit in the 2π-wedge billiard: the path drawn is a $(6, 1)$ orbit and a particle moving along it traces out the triangle twice (once in each direction) before it repeats, rather than once as it would in the plain circle.

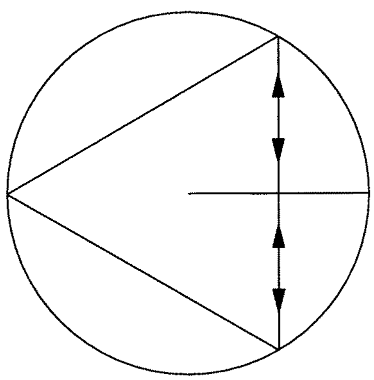

Figure 6. The 'triangle' orbit in the 2π-wedge billiard is twice as long as its equivalent in the circle.

5. Discussion

This paper has demonstrated formally that a factorial-over-power behaviour for the late terms of the Weyl series survives in the case of the wedge billiard, where the boundary is not smooth. Moreover, we have been able to uncover the higher-order corrections to the dominant asymptotic form. They are consistent with the extended conjecture (5).

The polygonal corner γ does not dominate the higher orders of the Weyl series, contributing only as a scaling factor in the circle memory terms. It does, however, generate the expected periodic orbit correction terms in the semiclassical expansion of the resolvent for real energies (via Stokes phenomena in term B (31)) together with a non-local whispering-gallery mode. The presence of the non-polygonal $\pi/2$ corners is felt at all orders through term A (30). A rigorous proof of these conjectures is still lacking.

It is clear that only one of the wedge orbits contributes to the asymptotics of the late terms. That this dominant orbit ($l = 2$) is not the shortest might not appear surprising in the context of the work of BH on 'two-bulge' (so-called 'bonio') billiards, but it should be noted that these were *concave* domains with C^∞ boundaries. The graphical selection mechanism proposed in that paper explained this apparent anomaly. Here we have a *convex* billiard, but with corners. We surmise that the corners contribute singularities in the analytic behaviour of the regularized resolvent which, at least, force the shorter orbits onto a different Riemann sheet of the path integral representation (Balian and Bloch 1972, 1974, Voros 1983). From the Darboux theorem (Dingle 1973), these shorter orbit singularities are then invisible at the leading asymptotic level of the higher orders of the Weyl series expansion. This is examined elsewhere (Howls and Trasler 1998).

Note that although the $l = 2$ orbit dominates the Weyl series, it does not dominate the periodic orbit corrections.

Clearly, since the shorter orbits have more than just two bounces, the graphical technique of BH for predicting the globally dominant orbit (cf Alonso and Gaspard 1994) is not possible for the wedge. Furthermore, with the apparent dependence on the Weyl series on the circle memory, there is clearly a need for a better selection mechanism via the singularity structure of the resolvent. This could come from a consideration of the functional equations satisfied by the resolvent but, as is known (Voros 1992), such equations are only possible on a case-by-case (or class-by-class) basis (quite apart from the problems associated with the accumulation of singularities at whispering-gallery modes).

The problem of the selection mechanism leads to an obvious question. Which is the more fundamental: the functional equations which determine the singularity structure, or the singularity structure from which the functional equations may be estimated or derived? The answer to that question will depend on the particular goal. If the late terms can be calculated, then a basic method for a limited determination of the path integral sheet structure is available from the hyperasymptotics of multiple integrals (Howls 1997).

Acknowledgments

One of us (SAT) acknowledges support from the EPSRC. CJH thanks the Royal Society, the JSPS and the Research Institute for Mathematical Sciences, Kyoto, Japan (where some of this work was carried out) for financial support.

Appendix A. The sum of Bessel K expansions (37)

We use the expansion (Abramowitz and Stegun 1972)

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z}\left(1 + \sum_{j=1}^{\infty} \frac{k_j}{z^j}\right) \tag{A.1}$$

to replace the Bessel functions in equation (37) where the coefficients k_j can be written

$$k_j = \frac{(1/2 + j)!}{2^j\, j!(1/2 - j)!}. \tag{A.2}$$

Formally, we have that

$$\frac{\gamma}{\pi l s} \sum_{\mu=1}^{\infty} \frac{K_1(ls\mu)}{\mu} \sim \frac{\gamma}{ls\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(1/2 + j)!}{j!(1/2 - j)!(2ls)^{j+1/2}}\, p(3/2 + j, \mathrm{e}^{-ls}) \tag{A.3}$$

where $p(n, x)$ is the polylogarithm (Wolfram 1996). We apply this result to (37) for $l = 2\gamma$, 2π. To leading order for $x < 1$ and $n > 1$,

$$p(n, x) \sim x. \tag{A.4}$$

(A.3) formally generates a series of exponential prefactors, whose dominant form is $\mathrm{e}^{-ls}s^{-3/2}$.

Appendix B. Asymptotic expansion of $I_0'(s)/sI_0(s)$

For $|\arg s| < \pi/2$ the large $|s|$ expansions of $I_0(s)$ and $I_0'(s)$ are (Abramowitz and Stegun 1972)

$$I_0(s) \sim \frac{\mathrm{e}^s}{\sqrt{2\pi s}} \sum_{n=0}^{\infty} \frac{d_n}{s^n} \qquad I_0'(s) \sim \frac{\mathrm{e}^s}{\sqrt{2\pi s}} \sum_{n=0}^{\infty} \frac{d_n'}{s^n} \tag{B.1}$$

where

$$d_n = \frac{(r-1/2)!^2}{2^r \pi r!} \qquad d_n' = -\frac{(4r^2-1)(r-3/2)!^2}{2^{r+2}\pi r!}. \tag{B.2}$$

The ratio of the Bessel functions can be written as a power series. Standard techniques lead to the recurrence relation

$$\frac{I_0'(s)}{I_0(s)} = \sum_{n=0}^{\infty} \frac{h_n}{s^n} \tag{B.3}$$

$$h_n = d_n' - \sum_{i=0}^{n-1} h_i d_{n-i} \qquad h_0 = 1. \tag{B.4}$$

Term A of (29) is equal to

$$-\frac{1}{4} \sum_{r=1}^{\infty} \frac{h_{r-1}}{s^r}. \tag{B.5}$$

Appendix C. Neville table algorithm

Our conjecture (5) gives an expansion of the Weyl coefficients over a large set of periodic orbits l_j. The leading-order contribution will arise from the smallest $|l_j|$ *in this set* (with index $j = 2$ in the wedge), so we postulate

$$c_r \sim \frac{(r-2)!}{l_2^r} \left\{ \alpha_0^{(2)} + \frac{\alpha_1^{(2)} l_2}{r-2} + \frac{\alpha_2^{(2)} l_2^2}{(r-2)(r-3)} + \cdots \right\} \qquad r \to \infty \tag{C.1}$$

and seek values for the $\alpha_k^{(2)}$. We follow Voros (1983) and first define A_r to denote the contents of the braces above, then

$$S_{r,1}^{(0)} = A_r \tag{C.2}$$

$$S_{r,p}^{(0)} = \frac{1}{p-1}\{(r-p)S_{r,p-1}^{(0)} - (r-2p+1)S_{r-1,p-1}^{(0)}\}. \tag{C.3}$$

Equations (C.2) and (C.3) will give us an approximation to the first coefficient we want to find, namely $\alpha_0^{(2)}$. For the higher-order corrections, we further define

$$S_{r,1}^{(k)} = \frac{r-k-1}{l}(S_{r,1}^{(k-1)} - \alpha_{k-1}^{(2)}) \tag{C.4}$$

$$S_{r,p}^{(k)} = \frac{1}{p-1}\{(r-p-k)S_{r,p-1}^{(k)} - (r-2p-k+1)S_{r-1,p-1}^{(k)}\}. \tag{C.5}$$

If (C.1) is correct, then at each level of recursion the approximations $S^{(0)}_{r,p}$ converge with the iterator p as

$$S^{(k)}_{r,p} = \alpha^{(2)}_k + \mathrm{O}\left(\frac{1}{r^p}\right). \tag{C.6}$$

There is a practical limit to the accuracy of this scheme, because we cannot take the limiting case of infinite r. In general, as p increases, the error in $S^{(k)}_{r,p}$ will only diminish so far before effects due to the other periodic orbits l_j become significant. The closer the next-smallest is to the dominating orbit, the less effective this algorithm is. In term A, this polluting influence is simply the first repetition of the 2-orbit and so is a factor $\mathrm{O}(2^{-r})$ smaller. Using *Mathematica* we perform all calculations in integer arithmetic, eliminating truncation errors.

Appendix D. Expanding the h_r

Making the $j = 0$ term from the sum in (B.4) explicit, we have the following definition for h_r:

$$h_r = -\frac{2(r-1/2)!^2}{2^r\pi r!} - \frac{(r-1/2)!(r-3/2)!}{2^r\pi r!} - \sum_{j=1}^{r-1}\frac{h_j(r-j-1/2)!^2}{2^{r-j}\pi(r-j)!}. \tag{D.1}$$

By inspection, we notice that the second term is of order r smaller than the first. The first and last terms in the j-sum are the same magnitude in r, of the same order as the second term of (D.1). This suggests that as we uncover each next-order term, we need to peel away terms from both ends of the j-sum.

Extracting these terms so that the sum over j runs from 2 to $r-2$, replacing h_{r-1} using (D.1) and rearranging that sum so it runs over the same values as the other, we have

$$\begin{aligned} h_r = &-\frac{(r-1/2)!^2}{2^{r-1}\pi r!} + \frac{3(r-3/2)!^2}{2^{r+1}\pi(r-1)!} - \frac{(r-5/2)!^2}{2^{r+2}\pi(r-2)!} - \frac{(r-1/2)!(r-3/2)!}{2^r\pi r!} \\ &+\frac{(r-3/2)!(r-5/2)!}{2^r\pi(r-1)!} - \sum_{j=2}^{r-2}\frac{h_j(r-j-1/2)!^2}{2^{r-j}\pi(r-j)!} \\ &\times\left\{1 - \frac{r-j}{4(r-j-1/2)^2}\right\}. \end{aligned} \tag{D.2}$$

Having found $h_1 = -1/2$, $h_2 = -1/8$ from (B.4), proceeding to the next level we eventually obtain

$$\begin{aligned} h_r = &-\frac{(r-1/2)!^2}{2^{r-1}\pi r!} + \frac{3(r-3/2)!^2}{2^{r+1}\pi(r-1)!} + \frac{11(r-5/2)!^2}{2^{r+4}\pi(r-2)!} + \frac{3(r-7/2)!^2}{2^{r+5}\pi(r-3)!} \\ &-\frac{(r-9/2)!^2}{2^{r+6}\pi(r-4)!} - \frac{(r-1/2)!(r-3/2)!}{2^r\pi r!} + \frac{(r-3/2)!(r-5/2)!}{2^{r+2}\pi(r-1)!} \\ &+\frac{(r-5/2)!(r-7/2)!}{2^{r+5}\pi(r-2)!} - \sum_{j=3}^{r-3}\frac{h_j(r-j-1/2)!^2}{2^{r-j}\pi(r-j)!} \\ &\times\left\{1 - \frac{r-j}{4(r-j-1/2)^2} - \frac{7(r-j)(r-j-1)}{32(r-j-1/2)^2(r-j-3/2)^2}\right\}. \end{aligned} \tag{D.3}$$

Now we make considerable use of the Stirling approximation (Abramowitz and Stegun 1972)

$$z! \sim \sqrt{2\pi}\,\exp\{-z + (z + \tfrac{1}{2})\ln z\} \qquad \text{as } z \to \infty \qquad |\arg z| < \pi/2. \tag{D.4}$$

The first term in (D.3) is at least $\mathrm{O}(r)$ larger than the rest, so we can identify the first term of h_{r-1} with that of (44). Thus, to leading order in r,

$$\frac{(r-3/2)!^2}{(r-1)!} \sim (r-2)!. \tag{D.5}$$

Starting with the hypothesis

$$\frac{(r-1/2)!^2}{r!} \sim (r-1)! + U_1(r-2)! + U_2(r-3)! + \cdots \tag{D.6}$$

based on the identification (D.5), we find that

$$U_1 = \lim_{r\to\infty} \frac{(r-1/2)!^2 - r!(r-1)!}{r!(r-2)!} = -\frac{1}{4} \tag{D.7}$$

after a little algebra. Continuing to the next stage,

$$U_2 = \lim_{r\to\infty} \frac{(r-1/2)!^2 - r!(r-1)! - U_1 r!(r-2)!}{r!(r-3)!} = \frac{9}{32}. \tag{D.8}$$

A similar result can be obtained starting with

$$\frac{(r-1/2)!(r-3/2)!}{r!} \sim (r-2)! + V_1(r-3)! + V_2(r-4)! + \cdots. \tag{D.9}$$

As above, the limit is taken and we obtain the value

$$V_1 = \lim_{r\to\infty} \frac{(r-1/2)!(r-3/2)! - r!(r-2)!}{r!(r-3)!} = -\frac{3}{4}. \tag{D.10}$$

That each limit is finite and non-zero suggests that the chosen hypotheses (D.6) and (D.9) are valid asymptotically. Substituting these for the ratios of factorials in (D.3), we find the leading-order behaviour of the coefficients h_r:

$$h_r \sim -\frac{(r-1)!}{2^{r-1}\pi} + \frac{(r-2)!}{2^r\pi} + \frac{3(r-3)!}{2^{r+2}\pi} + \frac{13(r-4)!}{2^{r+3}\pi} + \cdots \tag{D.11}$$

$$\Rightarrow \qquad c_r \sim \frac{1}{\pi}\frac{(r-2)!}{2^r} - \frac{1}{4\pi}\frac{(r-3)!}{2^{r-1}} - \frac{3}{32\pi}\frac{(r-4)!}{2^{r-2}} - \frac{13}{128\pi}\frac{(r-5)!}{2^{r-3}} - \cdots \tag{D.12}$$

in the resolvent (29).

References

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (Washington, DC: National Bureau of Standards)

Alonso D and Gaspard P 1994 *J. Phys. A: Math. Gen.* **27** 1599–607

Balian R and Bloch C 1972 *Ann. Phys.* **69** 76–160

——1974 *Ann. Phys.* **85** 514–45

Balian R, Parisi G and Voros A 1979 *Phys. Rev. Lett.* **41** 141–4

Baltes H P and Hilf E R 1976 *Spectra of Finite Systems* (Mannheim: B.I.-Wissenschaftsverlag)

Berry M V 1989 *Proc. R. Soc. Lond.* A **422** 7–21

Berry M V and Howls C J 1991 *Proc. R. Soc. Lond.* A **434** 657–75

——1994 *Proc. R. Soc. Lond.* A **447** 527–55

Bordag M, Elizalde E and Kirsten K 1996a *J. Math. Phys.* **37** 895–916

Bordag M, Geyer B, Kirsten K and Elizalde E 1996b *Comm. Math. Phys.* **179** 215–34

Bordag M, Kirsten K and Dowker S 1996c *Comm. Math. Phys.* **182** 371–93

Dietz B, Eckmann J P, Pillet C A, Smilansky U and Ussishkin I 1995 *Phys. Rev.* E **51** 4222–31

Dingle R B 1973 *Asymptotic Expansions: their Derivation and Interpretation* (New York and London: Academic)
Ecalle J 1981 *Les fonctions résurgentes* (3 vols) (Paris: Université de Paris-Sud)
——1984 Cinq applications des fonctions résurgentes *Preprint* 84T62 Orsay
Elizalde E, Lesduarte S and Romeo A 1993 *J. Phys. A: Math. Gen.* **26** 2409–19
Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York and London: Academic)
Howls C J 1992 *Proc. R. Soc. Lond.* A **439** 373–96
——1997 *Proc. R. Soc. Lond.* A **453** 2271–94
Howls C J and Trasler S A 1998 in preparation
Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** 173–8
Le Guillou J C and Zinn-Justin J (eds) 1990 *Large-order Behaviour of Perturbation Theory* (Amsterdam: North-Holland)
Lesduarte S and Romeo A 1994 *J. Phys. A: Math. Gen.* **27** 2483–95
Levitin M 1997 *Diff. Geom. Appl.* to appear
McKean H P Jr and Singer I M 1967 *J. Diff. Geom.* **1** 43–69
Olver F W J 1974 *Asymptotics and Special Functions* (New York and London: Academic)
Stewartson K and Waechter R T 1971 *Proc. Camb. Phil. Soc.* **69** 353–63
Voros A 1983 *Ann. Inst. H. Poincaré* A **39** 211–338
——1992 *Adv. Stud. Pure Math.* **21** 327–57
Waechter R T 1972 *Proc. Camb. Phil. Soc.* **72** 439–47
Watson G N 1948 *Theory of Bessel Functions* (Cambridge: Cambridge University Press)
Wolfram S 1996 *The Mathematica Book* (Cambridge: Wolfram Media, Inc. and Cambridge University Press)